Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

For six persons there appear to be two independent solutions, the one previously given by Dr. Judson, and the following:

ABCDEF,	ACEBDF,
ABDCFE,	ACBEFD,
ABEDFC,	ADECBF,
ABFECD,	ADBFCE,
ACDFBE,	AEDBCF.

192. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, O.

What is the difference between the squares of the two *infinite* continued fractions $\left(3+\frac{1}{6+\text{etc.}}\right)$ and $\left(2+\frac{1}{4+\text{etc.}}\right)$?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and L. E. NEWCOMB, Los Gatos, Cal.

Denote the value of the continued fractions by x and y.

Then $x-3=\dfrac{1}{6+x-3}=\dfrac{1}{x+3}$, $\therefore x^2-9=1$, $x^2=10$, $x=\sqrt{10}$;

$y-2=\dfrac{1}{4+y-2}=\dfrac{1}{y+2}$, $\therefore y^2-4=1$, $y^2=5$. $y=\sqrt{5}$.

$\therefore x^2-y^2=5=$required result.*

194. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In the determination of the canonical forms of Abelian transformations modulo p, one is led to the type $[b_1, b_2, b_3]$:

$$\xi_1'=\xi_1,\ \eta_1'=b_1\xi_1+\eta_1+b_2\xi_2+\eta_2+b_3\xi_3+\eta_3,\ \xi_2'=\xi_2-\xi_1,$$
$$\eta_2'=\eta_2+b_2\xi_2,\ \xi_3'=\xi_3-\xi_1,\ \eta_3'=\eta_3+b_3\xi_3.$$

Find its period and determine the conditions under which it is conjugate with $[c_1, c_2, c_3]$ under Abelian transformation.

Solution by PROPOSER.

By mathematical induction, we verify that the kth power of $[b_1, b_2, b_3]$ is

$$\xi_1'=\xi_1,\ \eta_1'=[kb_1-\tfrac{1}{6}k(k^2-1)(b_2+b_3)]\xi_1$$
$$+\eta_1+\tfrac{1}{2}k(k+1)(b_2\xi_2+b_3\xi_3)+k\eta_2+k\eta_3,$$
$$\xi_2'=\xi_2-k\xi_1,\ \eta_2'=\eta_2+kb_2\xi_2-\tfrac{1}{2}k(k-1)b_2\xi_1,$$
$$\xi_3'=\xi_3-k\xi_1,\ \eta_3'=\eta_3+kb_3\xi_3-\tfrac{1}{2}k(k-1)b_3\xi_1.$$

---

*Solutions based on the following interpretations are desirable. ED.

$$3+\cfrac{1}{6+\cfrac{1}{12+\cfrac{1}{24+\text{etc.}}}}\qquad 2+\cfrac{1}{4+\cfrac{1}{8+\cfrac{1}{12+\text{etc.}}}}\qquad 3+\cfrac{1}{6+\cfrac{1}{9+\cfrac{1}{12+\text{etc.}}}}\qquad 2+\cfrac{1}{4+\cfrac{1}{6+\cfrac{1}{12+\text{etc.}}}}$$

Hence if $p>3$, $[b_1, b_2, b_3]$ is of period p. For $p=2$ or 3, the period is p^2.

The question of conjugacy will be reduced to a discussion of simultaneous congruences. We assume that b_2 and b_3 are not both zero. We seek the linear substitutions S,

$$\xi_i'=\sum_{j=1}^{3}(\alpha_{ij}\xi_j+\gamma_{ij}\eta_j),\quad \eta_i'=\sum_{j=1}^{3}(\beta_{ij}\xi_j+\delta_{ij}\xi_j)\quad (i=1,2,3),$$

for which $[b_1, b_2, b_3]S=S[c_1, c_2, c_3]$. The conditions are

(1) $\gamma_{11}=\gamma_{12}=\gamma_{13}=\gamma_{21}=\gamma_{31}=0,\ \alpha_{12}+\alpha_{13}=\delta_{21}+\delta_{31}=0,$
$\delta_{2i}+\delta_{3i}=\delta_{11},\ c_2\gamma_{2i}=\delta_{21},$

(2) $b_i\gamma_{2i}=b_i\gamma_{3i}=-\alpha_{1i},\ b_i(\delta_{21}+\delta_{2i})=c_2\alpha_{2i},\ b_i(\delta_{31}+\delta_{3i})=c_3\alpha_{3i},\ c_3\gamma_{3i}=\delta_{31},$

(3) $b_1\delta_{i1}-\beta_{i2}-\beta_{i3}=c_i\alpha_{i1},\ c_1\alpha_{1i}+c_2\alpha_{2i}+c_3\alpha_{3i}+\beta_{2i}+\beta_{3i}=b_i(\delta_{11}+\delta_{1i}),$

(4) $\alpha_{22}+\alpha_{23}=\alpha_{32}+\alpha_{33}=\alpha_{11},\ c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31}$
$+\beta_{12}+\beta_{13}+\beta_{21}+\beta_{31}=b_1\delta_{11},$

where $i=2, 3$. Now c_2 and c_3 cannot both be zero, since this would require that all coefficients of η_1 in S should vanish, whereas its determinant $\neq 0$. Transforming by $(\xi_2\xi_3)(\eta_2\eta_3)$, if necessary, we may take $c_3\neq 0$, $b_3\neq 0$. For brevity we may exclude the special case $b_2=0$. Hence, by (1) and (2),

$$\delta_{21}=-c_2b_2^{-1}\alpha_{12}=c_2b_3^{-1}\alpha_{12}=c_3b_2^{-1}\alpha_{12}=-c_3b_3^{-1}\alpha_{12}.$$

If $\alpha_{12}\neq 0$, then $b_2=-b_3$, $c_2=-c_3$, a case here excluded. Hence $\alpha_{12}=0$, $\delta_{21}=0$. Then $\alpha_{13}=\delta_{31}=0$, and every $\gamma_{ij}=0$. By (2), $\delta_{2i}=b_i^{-1}c_2\alpha_{2i}$, $\delta_{3i}=b_i^{-1}c_3\alpha_{3i}$. *The substitutions S transforming $[b_1, b_2, b_3]$ into $[c_1, c_2, c_3]$ have therefore the matrix*

$$(5)\qquad \begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \alpha_{21} & 0 & \alpha_{22} & 0 & \alpha_{23} & 0 \\ \beta_{21} & 0 & \beta_{22} & b_2^{-1}c_2\alpha_{22} & \beta_{23} & b_3^{-1}c_2\alpha_{23} \\ \alpha_{31} & 0 & \alpha_{32} & 0 & \alpha_{33} & 0 \\ \beta_{31} & 0 & \beta_{32} & b_2^{-1}c_3\alpha_{32} & \beta_{33} & b_3^{-1}c_3\alpha_{33} \end{pmatrix},$$

subject only to conditions (3), (4), and $\delta_{2i}+\delta_{3i}=\delta_{11}$, the latter becoming now

$$(6)\qquad c_2\alpha_{2i}+c_3\alpha_{3i}=b_i\delta_{11}.$$

We next require that matrix (5) shall be Abelian (Jordan, *Traité*, p. 172; Dickson, *Linear Groups*, p. 89). The Abelian conditions all reduce to the following:

(7) $\alpha_{11}\delta_{11}=1,\ b_2^{-1}c_2\alpha_{22}\beta_{32}-b_2^{-1}c_3\alpha_{32}\beta_{22}+b_3^{-1}c_2\alpha_{23}\beta_{33}-b_3^{-1}c_3\alpha_{33}\beta_{23}=0,$

(8) $c_ib_2^{-1}\alpha_{i2}^2+c_ib_3^{-1}\alpha_{i3}^2=1,\ b_2^{-1}\alpha_{22}\alpha_{32}+b_3^{-1}\alpha_{23}\alpha_{33}=0,$

(9) $\delta_{11}\alpha_{i1}+\delta_{12}\alpha_{i2}+\delta_{13}\alpha_{i3}=0,\ \delta_{11}\beta_{i1}+\delta_{12}\beta_{i2}+\delta_{13}\beta_{i3}$
$-b_2^{-1}c_i\alpha_{i2}\beta_{12}-b_3^{-1}c_i\alpha_{i3}\beta_{13}=0,$
where $i=2$, 3. Adding the two equations (6) and applying $(4)_1$, we get $(c_2+c_3)\alpha_{11}=(b_2+b_3)\delta_{11}$. Then by $(7)_1$, $\delta_{11}^2=(c_2+c_3)/(b_2+b_3)$. This fraction must therefore be a quadratic residue modulo p. Suppose this condition satisfied, so that δ_{11} and α_{11} are determined except in sign. Then (6) and $(4)_1$ give

$$(10)\qquad \alpha_{23}=\alpha_{11}-\alpha_{22},\ c_3\alpha_{32}=b_2\delta_{11}-c_2\alpha_{22},\ c_3\alpha_{33}=b_3\delta_{11}-c_2\alpha_{11}+c_2\alpha_{22}.$$

Each of the conditions (8) then reduces to either of the forms

$$(11)\qquad (b_2^{-1}+b_3^{-1})\alpha_{22}^2-2b_3^{-1}\alpha_{11}\alpha_{22}+b_3^{-1}\alpha_{11}^2=c_2^{-1},\ [(b_2+b_3)\alpha_{22}-b_2\alpha_{11}]^2$$
$$=\frac{b_2+b_3}{c_2+c_3}\cdot\frac{b_2b_3c_3}{c_2}.$$

We have the further necessary condition that $b_2b_3c_2c_3$ shall be a quadratic residue modulo p. With this condition satisfied, α_{22} is determined by (11), and α_{23}, α_{32}, α_{33} by (10). There remains ten conditions (3), $(4)_2$, $(7)_2$, and (9) on the thirteen quantities δ_{12}, δ_{13}, α_{21}, α_{31}, β_{ij} $(i, j=1, 2, 3)$. By (6), we may write $(3)_2$ thus

$$(12)\qquad \beta_{2i}+\beta_{3i}=b_i\delta_{1i}.$$

Applying (6), (9) and (12), condition $(4)_2$ becomes

$$(13)\qquad b_2\delta_{12}^2+b_2\delta_{11}\delta_{12}+b_3\delta_{13}^2+b_3\delta_{11}\delta_{13}-2\delta_{11}(\beta_{12}+\beta_{13})-c_1+b_1\delta_{11}^2=0.$$

Since $\delta_{11}\neq 0$, we may employ (9) to determine α_{21}, α_{31}, β_{21}, β_{31} in terms of the remaining quantities. Conditions $(3)_1$ then become

$$(14)\qquad \beta_{i2}+\beta_{i3}=\alpha_{11}c_i(\delta_{12}\alpha_{i2}+\delta_{13}\alpha_{i3})\quad (i=2, 3).$$

One of these may be dropped since their sum is, by (6), the same as the sum of the pair (12). Then (6), and (14) for $i=2$, give

$$(15)\qquad \beta_{32}=b_2\delta_{12}-\beta_{22},\ \beta_{23}=\alpha_{11}c_2(\delta_{12}\alpha_{22}+\delta_{13}\alpha_{23})-\beta_{22},$$
$$\beta_{33}=b_3\delta_{13}+\beta_{22}-\alpha_{11}c_2(\delta_{12}\alpha_{22}+\delta_{13}\alpha_{23}).$$

Substituting these in $(7)_2$, we find that the coefficient of β_{22} is zero by (6), and that the remaining terms cancel. Hence the system of ten conditions is equivalent to the system (13), (15), (9). The latter may be solved in the order named. In particular, if $p\neq 2$, we may take δ_{1i}, α_{i1}, β_{2i}, β_{3i} $(i=2, 3)$ all zero and determine $\beta_{12}+\beta_{13}$ by (13), and β_{i1} by (9). Hence $[b_1, b_2, b_3]$ and $[c_1, c_2, c_3]$ are conjugate under Abelian transformation if and only if $(c_2+c_3)\div(b_2+b_3)$ and $b_2b_3c_2c_3$ are both quadratic residues modulo p.